Pergamon

Computers Math. Applic. Vol. 30, No. 3–6, pp. 197–206, 1995

0898-1221/95 $9.50 + 0.00

0898-1221(95)00097-6

Simultaneous Approximation from Convex Sets

J. SHI* AND R. HUOTARI
Department of Mathematics, Idaho State University, Pocatello, ID 83209, U.S.A.
shij@howland.isu.edu; huotari@isu.edu

Abstract—We characterize best simultaneous approximation from convex sets, in terms of one-sided Gateaux derivatives, using the ℓ^1 and ℓ^∞ vectorial norms, in a general setting and in L^1. In L^1, we give necessary and sufficient conditions for uniqueness. We show that p-type modulus of convexity implies order p-strong unicity of the best ℓ^∞ simultaneous approximation.

Keywords—Best approximation, Convex, Simultaneous, Gateaux derivative.

1. INTRODUCTION

This paper integrates two recent trends in approximation theory. A theme in the development of the non-linear theory is the attempt to extend the linear theory (in which the approximating set is a linear subspace of a normed linear space) to the case where the approximating set is only assumed to be convex. One of the goals of the present paper is to extend, in this way, several of the theorems in Allan Pinkus' influential book on L^1 approximation [1]. Another trend in approximation theory has been the study of vector-valued functions. A wide-ranging exposition of this field, also by Allan Pinkus, can be found in [2]. A special case of vector-valued approximation is simultaneous approximation, in which the best approximation vector is required to have the form $(u(x), \dots, u(x))$. The present paper is a study of the simultaneous approximation of two elements of a normed linear space from a set that is convex, but not necessarily linear. The particular spaces considered are L^1 and uniformly convex Banach spaces of power type p, $1 < p < \infty$ (which include the spaces L^p).

There are two common approaches to the characterization of best approximation. One is functional-analytic, and involves a theorem derived from the Hahn-Banach Theorem. The other is more classically analytic, and involves the use of the one-sided Gateaux derivative of the norm. These two approaches are efficiently motivated and described in Chapter 1 of [1], where the use of the Gateaux derivative is heuristically described as a "generalized perturbation technique." The present paper makes extensive use of the one-sided Gateaux derivative.

Suppose that $(X, \|\cdot\|)$ is a Banach space, and that K is a closed convex subset of X. Suppose f and g are fixed elements of X. With an appropriately defined vectorial norm $|||\cdot|||$, $(X \times X, |||\cdot|||)$ is a Banach space. For example, given a norm $\|\cdot\|_0$ on $\mathbb{R}^2$, let

$$|||(f,g)||| := \|(\|f\|, \|g\|)\|_0.$$

Then $|||\cdot|||$ is a norm on $X \times X$.

The necessary condition in Theorem 5.1 (in the case where the norm is smooth and the approximating set is a subspace) was made known to the authors by A. Egger. The authors also wish to acknowledge valuable conversations with M. Prophet.
*Current mailing address: 256 E. Helm Ave., #8, Salt Lake City, UT 84115

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Commonly used vectorial norms on $X \times X$ are defined as follows, for $(x, y) \in X \times X$,

$$
\begin{aligned}
|||(x,y)|||_p &= [\|x\|^p + \|y\|^p]^{1/p} \qquad 1 \le p < \infty, \\
|||(x,y)|||_\infty &= \max(\|x\|, \|y\|) \qquad p = \infty.
\end{aligned} \tag{i}
$$

Define a function $\Psi : X \to \mathbb{R}$ by $\Psi(k) = |||(f-k, g-k)|||$. An element $z \in K$ is said to be a *best simultaneous approximation* of f and g from K if $\Psi(z) = \inf\{\Psi(k) : k \in K\}$. If $||| \cdot |||_p$ is as defined in (i), we call z a *best ℓ^p-simultaneous approximation* of f and g from K. It is known that if X is a Banach space, then $X \times X$ with the ℓ^p vectorial norm is also a Banach space.

The following two theorems were early results in the line we are following. Theorem 1.1 was proved in [3] and Theorem 1.2, which was proved in [4], is a consequence of the Hahn-Banach Theorem.

THEOREM 1.1. *If X is a uniformly convex Banach space and K is a closed convex subset of X, then any two elements f and $g \in X$ have a unique best ℓ^∞-simultaneous approximation from K.*

THEOREM 1.2. *Let X be a normed linear space and let G be a subspace of X. Consider any two elements f and $g \in X$, such that $f \neq g \neq k (k \in \bar{G})$ and $g_0 \in G$. Then g_0 is a best ℓ^p-simultaneous approximation to f and g if and only if there exist elements $h_1, h_2 \in X^*$ with the properties*

$$
\begin{aligned}
\|h_1\|^q + \|h_2\|^q = 1, &\qquad \frac{1}{p} + \frac{1}{q} = 1, \\
h_1(g) + h_2(g) = 0, &\qquad \text{for all } g \in G,
\end{aligned}
$$

and

$$
h_1(f - g_0) + h_2(g - g_0) = \left[\|f - g_0\|^p + \|g - g_0\|^p\right]^{1/p}.
$$

One goal of this paper is to characterize the best ℓ^1- and ℓ^∞-simultaneous approximation of f and g by an element of K in terms of one-sided Gateaux derivatives. Compared to Theorem 1.2, the characterizations given below are more concrete because the Gateaux derivative is elementary. We will go on to consider characterization and uniqueness in the case where $X = L^1$. Finally, we will show that p-type modulus of convexity of the norm implies order p-strong uniqueness of the best ℓ^∞ simultaneous approximation. It is the authors' belief that much of the theory of individual approximation can be generalized to the simultaneous context using the characterization theorems given in the present paper.

2. BEST ℓ^1-SIMULTANEOUS APPROXIMATION

In this section, we characterize best ℓ^1-simultaneous approximation in terms of Gateaux derivatives. We assume that $(X, \|\cdot\|)$ is a Banach space. We first introduce the definition and some notations.

Let $f, g \in X$. If

$$
\lim_{t \to 0} \frac{\|f + tg\| - \|f\|}{t}
$$

exists, then the limit is said to be the *Gateaux derivative* of the norm at f in the direction g. Such limits do not necessarily exist. However the one-sided Gateaux derivatives always exist due to the following Lemma from Pinkus [1].

LEMMA 2.1. *Let $f, g \in X$, and set*

$$
r(t) = \frac{\|f + tg\| - \|f\|}{t}.
$$

On $(0, \infty)$, $r(t)$ is a non-decreasing function of t and is bounded below.

Hence for $f, g \in X$, if we set

$$
\tau_+(f, g) = \lim_{t \to 0^+} \frac{\|f + tg\| - \|f\|}{t},
$$

then it follows from Lemma 2.1 that $\tau_+(f, g)$ exists for every $f, g \in X$. The following theorem characterizes best ℓ^1-simultaneous approximation.

THEOREM 2.2. *Assume that K is a convex subset of X. Then w is a best ℓ^1-simultaneous approximation to f and g from K if and only if $\tau_+(f-w, w-h) + \tau_+(g-w, w-h) \geq 0$ for all $h \in K$.*

PROOF. Assume that w is a best approximation to f and g from K. By the definition

$$\Psi((1-t)w + th) \geq \Psi(w),$$

i.e.,

$$\|f - w + t(w-h)\| + \|g - w + t(w-h)\| \geq \|f - w\| + \|g - w\|$$

for every $h \in K$ and $0 \leq t \leq 1$. Hence letting $t \to 0$, we have

$$\tau_+(f-w, w-h) + \tau_+(g-w, w-h) \geq 0$$

for all $h \in K$.

Conversely, assume $\tau_+(f-w, w-h) + \tau_+(g-w, w-h) \geq 0$ for all $h \in K$. From Lemma 2.1, $r(t)$ is a nondecreasing function of t on $(0,\infty)$. Setting $t = 1$, we have

$$\|f - h\| + \|g - h\| - \|f - w\| - \|g - w\| \geq \tau_+(f-h, w-h) + \tau_+(g-h, w-h) \geq 0.$$

Thus, $\Psi(h) \geq \Psi(w)$ for all $h \in K$. ∎

It is apparent that Theorem 2.2 can be extended to the best ℓ^1-simultaneous approximation of more than two elements.

3. BEST ℓ^1-SIMULTANEOUS APPROXIMATION IN L^1

In the present section, we restrict our attention to the case where $X = L^1$. Suppose that C is a set, Σ is a σ-field of subsets of C, and μ is a positive measure on Σ. By $L^1(C,\mu)$, we denote the set of all real valued, μ-integrable functions, and we let

$$\|f\|_1 := \int_C |f(x)|\, d\mu(x).$$

Then $L^1(C,\mu)$, with the norm $\|\cdot\|_1$, is a Banach space.

For each $f \in L^1(C,\mu)$, let $Z(f) = \{x | f(x) = 0\}$ and $N(f) = C \backslash Z(f)$. We define

$$\operatorname{sgn}(f(x)) = \begin{cases} 1, & f(x) > 0, \\ 0, & f(x) = 0, \\ -1, & f(x) < 0. \end{cases}$$

Using Theorem 2.2, we can easily generalize a theorem stated by Pinkus (Theorem 2.1 in [1]) characterizing L^1-approximation. We note that Deutsch (page 100 in [5]) gave a functional-analytic proof of this theorem when the approximating set is convex. The following theorem is a generalization of this result from individual to simultaneous approximation. Its proof uses the one-sided Gateaux derivative.

THEOREM 3.1. *Let K be a closed convex set and $f, g \in L^1(C,\mu) \setminus K$. Then w is a best ℓ^1-simultaneous approximation to f and g from K if and only if*

$$\int_C (\operatorname{sgn}(f-w) + \operatorname{sgn}(g-w))\,(h-w)\, d\mu \leq \left(\int_{Z(f-w)} + \int_{Z(g-w)} \right) |h - w|\, d\mu$$

for all $h \in K$.

PROOF. By definition

$$\tau_+(f-w, w-h) = \lim_{t\to 0^+} \frac{\|f-w+t(w-h)\|_1 - \|f-w\|_1}{t}.$$

For $t > 0$,

$$\begin{aligned}\frac{\|f-w+t(w-h)\|_1 - \|f-w\|_1}{t} &= \frac{1}{t}\int_C (|f-w+t(w-h)| - |f-w|)\,d\mu \\ &= \int_{Z(f-w)} |h-w|\,d\mu \\ &\quad + \frac{1}{t}\int_{N(f-w)} (|f-w+t(h-w)| - |f-w|)\,d\mu.\end{aligned}$$

On $N(f-w)$

$$\left|\frac{|f-w+t(w-h)| - |f-w|}{t}\right| \le |w-h|$$

and

$$\begin{aligned}\frac{|f-w+t(w-h)| - |f-w|}{t} &= \frac{|f-w+t(w-h)|^2 - |f-w|^2}{t(|f-w+t(w-h)| + |f-w|)} \\ &= \frac{2(f-w)(w-h) + t|h-w|^2}{|f-w+t(w-h)| + |f-w|}.\end{aligned}$$

Thus on $N(f-w)$,

$$\lim_{t\to 0^+} \frac{|f-w+t(w-h)| - |f-w|}{t} = \frac{2(f-w)(w-h)}{2|f-w|} = \operatorname{sgn}(f-w)(w-h).$$

Applying Lebesgue's Dominated Convergence Theorem, we obtain that

$$\begin{aligned}\tau_+(f-w, w-h) &= \int_{Z(f-w)} |w-h|\,d\mu + \int_C \operatorname{sgn}(f-w)(w-h)\,d\mu \\ &= \int_{Z(f-w)} |w-h|\,d\mu - \int_C \operatorname{sgn}(f-w)(h-w)\,d\mu.\end{aligned}$$

Similarly,

$$\begin{aligned}\tau_+(g-w, w-h) &= \int_{Z(g-w)} |w-h|\,d\mu + \int_C \operatorname{sgn}(g-w)(w-h)\,d\mu \\ &= \int_{Z(g-w)} |w-h|\,d\mu - \int_C \operatorname{sgn}(g-w)(h-w)\,d\mu.\end{aligned}$$

Therefore, the theorem follows from Theorem 2.2. This completes the proof. ∎

It is apparent that Theorem 3.1 can be extended to the best ℓ^1-simultaneous approximation of more than two elements.

We now turn our attention to the uniqueness of the ℓ^1-simultaneous L^1 approximation. The following lemma generalizes the necessary condition of Proposition 2.4 in [1] in two directions: from a linear approximating set to one which is only assumed to be convex; and from individual to simultaneous approximation.

LEMMA 3.2. *Let K be a convex subset of $L^1(C,\mu)$ and $f,g \in L^1(C,\mu)$. Assume w is a best ℓ^1-simultaneous approximation of f and g from K. If $w' \in K$, $w' \neq w$, is also a best ℓ^1-simultaneous approximation of f and g from K, then*

$$\int_C (\operatorname{sgn}(f-w)+\operatorname{sgn}(g-w))\,(w'-w)\,d\mu = \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right)|w'-w|\,d\mu.$$

PROOF. Note that

$$\begin{aligned}
\|f-w\|_1+\|g-w\|_1 &= \int_C \operatorname{sgn}(f-w)(f-w)\,d\mu + \int_C \operatorname{sgn}(g-w)(g-w)\,d\mu \\
&= \int_C \operatorname{sgn}(f-w)(f-w')\,d\mu + \int_C \operatorname{sgn}(f-w)(w'-w)\,d\mu \\
&\quad + \int_C \operatorname{sgn}(g-w)(g-w')\,d\mu + \int_C \operatorname{sgn}(g-w)(w'-w)\,d\mu.
\end{aligned} \tag{i}$$

Since w is a best simultaneous approximation, Theorem 3.1 implies that

$$\int_C (\operatorname{sgn}(f-w)+\operatorname{sgn}(g-w))\,(w'-w)\,d\mu \le \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right)|w'-w|\,d\mu. \tag{ii}$$

By (i) and (ii),

$$\begin{aligned}
\|f-w\|_1+\|g-w\|_1 &\le \int_C \operatorname{sgn}(f-w)(f-w')\,d\mu + \int_C \operatorname{sgn}(g-w)(g-w')\,d\mu \\
&\quad + \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right)|w'-w|\,d\mu \\
&= \int_{N(f-w)} \operatorname{sgn}(f-w)(f-w')\,d\mu + \int_{N(g-w)} \operatorname{sgn}(g-w)(g-w')\,d\mu \\
&\quad + \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right)|w'-w|\,d\mu \\
&\le \int_{N(f-w)} |f-w'|\,d\mu + \int_{N(g-w)} |g-w'|\,d\mu \\
&\quad + \int_{Z(f-w)} |f-w'| + \int_{Z(g-w)} |g-w'|\,d\mu \\
&= \|f-w'\|_1 + \|g-w'\|_1.
\end{aligned} \tag{iii}$$

Since w and w' are each best simultaneous approximations of f and g, $\|f-w\|_1+\|g-w\|_1 = \|f-w'\|_1+\|g-w'\|_1$. Hence, (i) and (iii) imply that

$$\begin{aligned}
&\int_{N(f-w)} \operatorname{sgn}(f-w)(f-w')\,d\mu + \int_{N(g-w)} \operatorname{sgn}(g-w)(g-w')\,d\mu + \int_C (\operatorname{sgn}(f-w)+\operatorname{sgn}(g-w))\,(w'-w)\,d\mu \\
&\qquad = \int_{N(f-w)} |f-w'|\,d\mu + \int_{N(g-w)} |g-w'|\,d\mu + \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right)|w'-w|\,d\mu
\end{aligned}$$

Since $\operatorname{sgn}(f-w)(f-w') \leq |f-w'|$ and $\operatorname{sgn}(g-w)(g-w') \leq |g-w'|$, it follows from (ii) that

$$\int_C (\operatorname{sgn}(f-w) + \operatorname{sgn}(g-w))\,(w'-w)\,d\mu = \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right) |w'-w|\,d\mu$$

which completes the proof of Lemma 3.2. ∎

From Lemma 3.2, we obtain a sufficient condition for the uniqueness of the best ℓ^1-simultaneous L^1 approximation, which generalizes Corollary 2.5 in [1].

THEOREM 3.3. *Let K be a convex subset of $L^1(C,\mu)$ and w a best ℓ^1-simultaneous approximation of f and g from K. Assume that*

$$\int_C (\operatorname{sgn}(f-w) + \operatorname{sgn}(g-w))\,(h-w)\,d\mu < \left(\int_{Z(f-w)} + \int_{Z(g-w)}\right) |h-w|\,d\mu$$

for all $h \in K$, $h \neq w$. Then the best ℓ^1-simultaneous approximation of f and g from K is unique.

4. BEST ℓ^1-SIMULTANEOUS APPROXIMATION IN ℓ^1_n

In the present section, we specialize Section 3 to the case where the measure μ is purely atomic. Let $C := \{1,\dots,n\}$. Then approximation in $L^1(C,\mu)$ corresponds to approximation in the vector space $\mathbb{R}^n$.

For greater generality, we consider an arbitrarily weighted ℓ_1^n norm. Let

$$T = \{t : t = (t_1,\cdots,t_n), t_i > 0, i = 1,\cdots,n\}.$$

Every $t \in T$ is called a *weight*. On R^n, we define the $\ell^1_n(t)$-norm given by

$$\|x\|_t = \sum_{i=1}^n |x_i| t_i,$$

where $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$. Thus, the context of the present section is $L^1(C,\mu)$, where $C := \{1,\dots,n\}$ and μ is the "counting" measure corresponding to the weight t. We shall show that the result of Theorem 3.3 is in fact a characterization for uniqueness of best simultaneous approximation in the space $\ell^1_n(t)$.

An application of Theorem 3.1 yields the following theorem.

THEOREM 4.1. *Let K be a closed convex set and $f, g \in \mathbb{R}^n \setminus K$. Then w is a best ℓ^1-simultaneous approximation to f and g from K in the $\ell^1_n(t)$-norm if and only if*

$$\sum_{i=1}^n (\operatorname{sgn}(f_i-w_i) + \operatorname{sgn}(g_i-w_i))\,(h_i-w_i)t_i \leq \left(\sum_{i\in Z(f-w)} + \sum_{i\in Z(g-w)}\right) |h_i-w_i|t_i$$

for all $h \in K$.

An application of Theorem 3.3 yields.

THEOREM 4.2. *Let K be a convex subset of $\mathbb{R}^n$ and w be a best ℓ^1-simultaneous approximation of f and g from K in the $\ell^1_n(t)$-norm. Assume that*

$$\sum_{i=1}^n (\operatorname{sgn}(f_i-w_i) + \operatorname{sgn}(g_i-w_i))\,(h_i-w_i)t_i < \left(\sum_{i\in Z(f-w)} + \sum_{i\in Z(g-w)}\right) |h_i-w_i|t_i$$

for all $h \in K$, $h \neq w$. Then the best ℓ^1-simultaneous approximation of f and g from K is unique.

The following lemma is a generalization of Lemma 6.4 in [1].

LEMMA 4.3. *Let K be a convex subset of $\mathbb{R}^n$, $f, g \in \mathbb{R}^n \setminus K$, and $t \in T$. Let w be a best ℓ^1-simultaneous approximation of f and g from K in the $\ell_n^1(t)$-norm. Assume that $h \in K, h \neq w$, and*

$$\sum_{i=1}^{n} (\operatorname{sgn}(f_i - w_i) + \operatorname{sgn}(g_i - w_i))\,(h_i - w_i)t_i = \left(\sum_{i \in Z(f-w)} + \sum_{i \in Z(g-w)} \right) |h_i - w_i| t_i. \tag{i}$$

Then there exists an $\epsilon_0 > 0$, such that for every $\epsilon \in [0, \epsilon_0)$, $(1-\epsilon)w + \epsilon h$ is a best ℓ^1-simultaneous approximation of f and g from K.

PROOF. Since C is finite, there exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in [0, \epsilon_0)$,

$$\operatorname{sgn}(f_i - w_i) = \operatorname{sgn}(f_i - w_i + \epsilon(w_i - h_i)) \tag{ii}$$

for every $i \in N(f-w)$, and

$$\operatorname{sgn}(g_i - w_i) = \operatorname{sgn}(g_i - w_i + \epsilon(w_i - h_i)) \tag{iii}$$

for every $i \in N(g-w)$. Thus, for every such ϵ,

$$\begin{aligned}
&\|f - w + \epsilon(w-h)\|_t + \|g - w + \epsilon(w-h)\|_t - \|f-w\|_t - \|g-w\|_t \\
&= \sum_{i=1}^{n} \operatorname{sgn}(f_i - w_i + \epsilon(w_i - h_i))(f_i - w_i + \epsilon(w_i - h_i))t_i \\
&\quad + \sum_{i=1}^{n} \operatorname{sgn}(g_i - w_i + \epsilon(w_i - h_i))(g_i - w_i + \epsilon(w_i - h_i))t_i \\
&\quad - \sum_{i=1}^{n} \operatorname{sgn}(f_i - w_i)(f_i - w_i)t_i - \sum_{i=1}^{n} \operatorname{sgn}(g_i - w_i)(g_i - w_i)t_i \\
&= \epsilon \sum_{i=1}^{n} \operatorname{sgn}(f_i - w_i)(w_i - h_i)t_i + \epsilon \sum_{i \in Z(f-w)} |w_i - h_i| t_i \\
&\quad + \epsilon \sum_{i=1}^{n} \operatorname{sgn}(g_i - w_i)(w_i - h_i)t_i + \epsilon \sum_{i \in Z(g-w)} |w_i - h_i| t_i \\
&= -\epsilon \sum_{i=1}^{n} \operatorname{sgn}(f_i - w_i)(h_i - w_i)t_i + \epsilon \sum_{i \in Z(f-w)} |h_i - w_i| t_i \\
&\quad - \epsilon \sum_{i=1}^{n} \operatorname{sgn}(g_i - w_i)(h_i - w_i)t_i + \epsilon \sum_{i \in Z(g-w)} |h_i - w_i| t_i = 0
\end{aligned}$$

where the second equality follows from (ii) and (iii), and the last from (i).

Therefore, $(1-\epsilon)w + \epsilon h$ is a best ℓ^1-simultaneous approximation of f and g from K. ∎

Based on Lemma 4.3 and Theorem 4.2, we obtain the following theorem characterizing the uniqueness of the best ℓ^1-simultaneous approximation in the $\ell_n^1(t)$-norm. It generalizes Theorem 6.5 in [1].

THEOREM 4.4. *Let K be a convex subset of $\mathbb{R}^n$, $f, g \in \mathbb{R}^n \setminus K$. Then w is the unique best ℓ^1-simultaneous approximation of f and g from K in the $\ell_n^1(t)$-norm if and only if*

$$\sum_{i=1}^{n} (\operatorname{sgn}(f_i - w_i) + \operatorname{sgn}(g_i - w_i))\,(h_i - w_i)t_i < \left(\sum_{i \in Z(f-w)} + \sum_{i \in Z(g-w)} \right) |h_i - w_i| t_i$$

for all $h \in K$, $h \neq w$.

It is apparent that Theorem 4.4 can be extended to the best ℓ^1-simultaneous approximation of more than two elements.

5. BEST ℓ^∞-SIMULTANEOUS APPROXIMATION

In this section, we assume that $(X, \|\cdot\|)$ is a uniformly convex Banach space, and that K is a closed convex set of X as usual. It is known from Theorem 1.1 that the best ℓ^∞-simultaneous approximation exists and is unique. We will denote it by $(fg)^*$. We will denote by f^* (resp., g^*) the unique best individual $\|\cdot\|$-approximation of f (resp., g) from K.

Amir and Ziegler [6] discovered a necessary condition for best ℓ^∞-simultaneous approximation. A loose statement of the Amir-Ziegler condition is that $(fg)^*$ is either a *relative midpoint* of f and g (i.e., $\|f-(fg)^*\| = \|g-(fg)^*\|$) or $(fg)^*$ is either f^* or g^*. Huotari and Sahab [7] discussed the case where X is smooth and K is a subspace of X, and obtained a characterization of the best ℓ^∞-simultaneous approximation of f and g. The next theorem assumes neither the smoothness of the norm, nor the linearity of K.

THEOREM 5.1. *In the above context, consider the following conditions:*

(1) *For every $h \in K$, either $\tau_+(f-w, w-h) \geq 0$ or $\tau_+(g-w, w-h) \geq 0$;*
(2) $\|f-w\| = \|g-w\|$;
(3) $\|g-w\| < \|f-w\|$ *and* $w = f^*$;
(4) $\|f-w\| < \|g-w\|$ *and* $w = g^*$.

Then $w = (fg)^*$ if and only if (1) and exactly one of (2), (3), or (4) hold.

PROOF. Suppose first that $w = (fg)^*$. If w does not satisfy (1), then there exists $h \in K$ such that $\tau_+(f-w, w-h) < 0$ and $\tau_+(g-w, w-h) < 0$. Then by the definition of the Gateaux derivative,

$$\begin{aligned} \|f-((1-\beta)w+\beta h)\| &< \|f-w\| \qquad \text{and} \\ \|g-((1-\beta)w+\beta h)\| &< \|g-w\| \end{aligned}$$

for some $\beta \in (0,1)$.

So,

$$\Psi((1-\beta)w+\beta h) < \Psi(w).$$

This is a contradiction.

If neither (2), (3), nor (4) holds, then either

$$\begin{aligned} \|g-w\| &< \|f-w\| \quad (\text{and} \quad w \neq f^*) \quad \text{or} \\ \|f-w\| &< \|g-w\| \quad (\text{and} \quad w \neq g^*). \end{aligned}$$

Since the two cases are treated similarly, we suppose the second. Since $w \neq g^*$ and X is strictly convex, we have

$$\|g-((1-\beta)w+\beta g^*)\| < \|g-w\|$$

for every $\beta \in (0,1)$. And since $\|f-w\| < \|g-w\|$ and every norm is continuous,

$$\|f-((1-\beta_0)w+\beta_0 g^*)\| < \|g-w\|$$

for some $\beta_0 \in (0,1)$. Therefore, $\Psi((1-\beta_0)w+\beta_0 g^*) < \Psi(w)$ for some $\beta_0 \in (0,1)$, a contradiction.

For the converse, suppose that (1) and (2) hold. For an arbitrary $h \in K$, either $\tau_+(f-w, w-h) \geq 0$ or $\tau_+(g-w, w-h) \geq 0$. Without loss of generality, we suppose the first. Now if we let

$$\gamma(t) = \frac{\|f-w+t(w-h)\| - \|f-w\|}{t},$$

then $\gamma(t)$ is a nondecreasing function of t on $(0,\infty)$ by Lemma 2.1. Setting $t=1$, we have

$$\|f-h\| - \|f-w\| \geq \tau_+(f-w, w-h) \geq 0.$$

Thus, $\|f - w\| \le \|f - h\|$. Now $\Psi(w) = \max(\|f - w\|, \|g - w\|) = \|f - w\| \le \|f - h\| \le \max(\|f - h\|, \|g - h\|) = \Psi(h)$. Since h was arbitrary, w is a best simultaneous approximation of f and g. It follows from the uniqueness of $(fg)^*$ that $w = (fg)^*$.

Suppose that (3) holds and $h \in K$. Then $w = f^*$ implies that $\|f - w\| \le \|f - h\|$. Hence, $\Psi(w) = \max(\|f - w\|, \|g - w\|) = \|f - w\| \le \|f - h\| \le \max(\|f - h\|, \|g - h\|) = \Psi(h)$, whence $w = (fg)^*$ by the uniqueness of $(fg)^*$.

A similar argument applies if (4) holds. ∎

Inspection of the above proof shows that the following is also true: If (3) or (4) holds, then $w = (fg)^*$.

Let us turn our attention now to the problem of strong unicity of best ℓ^∞ simultaneous approximation. Let $S = S(X)$ be the unit sphere, $\{x \in X : \|x\| = 1\}$, and $B = B(X)$ be the closed unit ball, $\{x \in X : \|x\| \le 1\}$. The *modulus of convexity* of X is a function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S, \|x - y\| \ge \epsilon\right\}.$$

In some sense, the modulus of convexity measures the flattest spot on the unit ball. For $p > 1$, we say that X has modulus of convexity of *power type p* if there exists $k \in (0, \infty)$ such that $\delta(\epsilon) \ge k\epsilon^p$. An example of such a space is $L^p[0,1], 1 < p < \infty$, which has modulus of convexity of power type $\max(2, p)$ [8].

Diestel [9] (Lemma 4, p. 125) showed that

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B, \|x - y\| \ge \epsilon\right\}.$$

This characterization of δ is sometimes more convenient than is the definition, and will be used in the proof of our next theorem.

Suppose that $f \in X$ is fixed. We say that f^* is *strongly unique of order* α if, for some $M = M(f) > 0$, there exists a $\gamma = \gamma(f, M) > 0$ such that, for all $h \in K$ with $\|h - f^*\| \le M$,

$$\|h - f\| \ge \|f^* - f\| + \gamma\|h - f^*\|^\alpha.$$

In the context of simultaneous approximation, we say that $(fg)^*$ is strongly unique of order α if, for some $M > 0$, there exists a $\gamma = \gamma(f, g, M) > 0$, such that, for all $h \in K$ with $\|h - (fg)^*\| \le M$,

$$\Psi(h) \ge \Psi((fg)^*) + \gamma\|h - (fg)^*\|^\alpha.$$

The following theorem, published by Lin [10], describes a connection between modulus of convexity and strong uniqueness.

THEOREM 5.2. *Suppose X is a uniformly convex Banach space with modulus of convexity of power type p, K is a closed convex subset of X, and $f \in X$. Then f^* is strongly unique of order p.*

Assuming linearity of K, Huotari and Sahab [7] showed that Lin's theorem is essentially a characterization in the individual-approximation context. Also in [7], it was shown that Lin's theorem can be extended to the simultaneous-approximation context if K is assumed to be linear and the norm is assumed to be smooth. The following theorem makes neither of these two assumptions.

THEOREM 5.3. *Suppose X is a uniformly convex Banach space with modulus of convexity of power type p, and K is a closed convex subset of X. Then, for any $f, g \in X$, $(fg)^*$ is strongly unique of order p.*

PROOF. Suppose first that $(fg)^* = g^*$. By Theorem 5.2, there exist $M > 0$ and $\gamma > 0$ such that

$$\|g - h\| \ge \|(fg)^* - g\| + \gamma\|h - (fg)^*\|^p \tag{i}$$

whenever $\|h-(fg)^*\| \le M$. By Theorem 5.1, $\|g-(fg)^*\| \ge \|f-(fg)^*\|$. Thus, (i) implies that

$$\Psi(h) \ge \Psi((fg)^*) + \gamma\|h-(fg)^*\|^p.$$

A symmetric argument applies if $(fg)^* = f^*$.

For the remainder of this proof, we will suppose that $f^* \ne (fg)^* \ne g^*$. In this case, Theorem 5.1 entails that $\|f-(fg)^*\| = \|g-(fg)^*\|$ and for every $h \in K$, either $\tau_+(f-(fg)^*, (fg)^*-h) \ge 0$ or $\tau_+(g-(fg)^*, (fg)^*-h) \ge 0$. Without loss of generality, we assume $\tau_+(f-(fg)^*, (fg)^*-h) \ge 0$.

For every $h \in K$ with $\|h-(fg)^*\| \le 2\|f-(fg)^*\|$, it follows from the definition of the one-sided Gateaux derivative and the convexity of the norm that

$$3\|f-(fg)^*\| \ge \|f-h\| \ge \left\| f - \frac{(fg)^*+h}{2} \right\| \ge \|f-(fg)^*\|.$$

Given $z \in X$, let $n(z) = z/\|f-h\|$. By Diestel's Lemma cited above, there exists $k > 0$, such that

$$1 - \frac{\|n(f-h)+n(f-(fg)^*)\|}{2} \ge k\|n(f-h)-n(f-(fg)^*)\|^p.$$

So,

$$\left\| f - \frac{h+(fg)^*}{2} \right\| \le \left(1 - k\frac{\|(fg)^*-h\|^p}{\|f-h\|^p} \right) \|f-h\|.$$

Therefore,

$$\begin{aligned} \|f-h\| &\ge \left\| f - \frac{h+(fg)^*}{2} \right\| + k\frac{\|(fg)^*-h\|^p}{\|f-h\|^{p-1}} \\ &\ge \|f-(fg)^*\| + \gamma\|(fg)^*-h\|^p, \end{aligned}$$

where

$$\gamma = \frac{k}{3^{p-1}\|f-(fg)^*\|^{p-1}}.$$

Since $\|f-(fg)^*\| = \|g-(fg)^*\|$, we have that

$$\begin{aligned} \Psi(h) &= \max(\|f-h\|, \|g-h\|) \\ &\ge \|f-h\| \\ &\ge \|f-(fg)^*\| + \gamma\|(fg)^*-h\|^p \\ &= \max(\|f-(fg)^*\|, \|g-(fg)^*\|) + \gamma\|(fg)^*-h\|^p \\ &= \Psi((fg)^*) + \gamma\|(fg)^*-h\|^p. \end{aligned}$$

This completes the proof. ∎

REFERENCES

1. A. Pinkus, *On L^1 Approximation*, Cambridge University Press, (1989).
2. A. Pinkus, Uniqueness in vector-valued approximation, *J. Approx. Theory* **73**, 17–92, (1993).
3. D.S. Goel, A.S.B. Holland, C. Nasim, and B.N. Sahney, On best simultaneous approximation in normed linear spaces, *Canad. Math. Bull.* **17**, 523–527, (1974).
4. D.S. Goel, A.S.B. Holland, C. Nasim, and B.N. Sahney, Characterization of an element of best ℓ^p-simultaneous approximation, In *S. Ramanujan Memorial Volume*, pp. 10–14, Madras, (1974).
5. F. Deutsch, *Applications of Functional Analysis to Approximation Theory*, Lecture Notes, (1992).
6. D. Amir and Z. Ziegler, Relative Chebyshev centers in normed linear spaces I, *J. Approx. Theory* **29**, 235–252, (1980).
7. R. Huotari and S. Sahab, Strong unicity versus modulus of convexity, *Bull. Austr. Math. Soc.* (to appear).
8. J. Diestel, *Geometry of Banach Spaces—Selected Topics*, LNM 485, Springer-Verlag, (1975).
9. J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, (1984).
10. P.K. Lin, Strongly unique best approximation in uniformly convex Banach spaces, *J. Approx. Theory* **56**, 101–107, (1989).
11. A. Egger and G.D. Taylor, Local strong uniqueness, *J. Approx. Theory* **58**, 268–280, (1989).